Generalisations of the Tits representation

Daan Krammer

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Abstract

We construct a group K_n with properties similar to infinite Coxeter groups. In particular, it has a geometric representation featuring hyperplanes and simplicial chambers. The generators of K_n are given by 2-element subsets of $\{0, \dots, n\}$. We give some easy combinatorial results on the finite residues of K_n.

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1 Introduction

A *Coxeter group* is a group W presented with generating set S and relations s^2 for all $s \in S$ and at most one relation $(st)^{m(s,t)}$ for every pair $\{s, t\} \subset S$ (where $m(s, t) = m(t, s)$). It is known that then the natural map $S \to W$ is injective; we think of it as an inclusion. We call the pair (W, S) a *Coxeter system.*

We generalise this as follows. For any set S, let F_S denote the free monoid on S. A *fully coloured graph* is a triple (V, S, m) where V, S are sets, $m\colon V \times S \times S \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ is a map, and an *action* $V \times F_S \to V$ written $(v, g) \mapsto vg$ is specified, satisfying the following.

- For all $v \in V$, $s \in S$ we have $(vs)s = v$.
- Let $v \in V$, $s, t \in S$. Then $m(v; s, t) = 1$ if and only if $s = t$. Moreover $m(v; s, t) = m(v; t, s)$ and $m(v; s, t) = m(vs; s, t)$. Also, if $k := m(v; s, t)$ is finite then $v(st)^k = v$.
- The set V is universal. That is, let (V', S, m') satisfy the above too and let $f\colon V' \to V$ be a map satisfying (i) $(fv)s = f(vs)$ for all v, s; (ii) $m'(v; s, t) = m(fv; s, t)$ for all v, s, t. Then the restriction of f to every F_S-orbit in V' is injective.

Every Coxeter system (W, S) gives rise to a *Coxeter fully coloured graph* (W, S, m) where one defines $m(w; s, t)$ to be the order of st and the action $W \times S \to W$ to be multiplication.

Let (V, S, m) be a fully coloured graph. For $I \subset S$, an *I-residue* is a subset of V of the form $\{vg \mid g \in F_I\}$. Contrary to the Coxeter case, it may happen that an $\{s, t\}$-residue $R \subset V$ containing v is such that $\#R \neq 2m(v; s, t)$; see remark 5(b) for an example of this.

A *generalised simplicial hyperplane arrangement* (gsa) consists of an open convex cone U^0 in a finite dimensional real vector space Q together with a set

$\mathcal{A}$ of hyperplanes[1] which is locally finite[2] in U^0 and such that every (closed) chamber[3] is simplicial[4] and such that for every chamber C and every boundary point $c \in \partial C$ there is a chamber $D \neq C$ containing c. For two chambers C, D, let $d(C, D)$ be the number of hyperplanes $H \in \mathcal{A}$ separating C^0 from D^0 (0 is relative interior). A *cell* is an intersection of chambers (we use a different but equivalent definition in the main text). If $U^0 = Q$ then $\mathcal{A}$ is finite and we obtain a simplicial hyperplane arrangement.

Every gsa $\mathcal{A}$ gives rise to a fully coloured graph $G(\mathcal{A}) = (V, S, m)$ which is defined by the following. We let V be the set of chambers. Let n denote the codimension of the smallest cell. There is a unique equivalence relation on the set of $(n-1)$-codimensional cells of n equivalence classes such that no two equivalent $(n-1)$-codimensional cells are in one chamber. Let S be the set of equivalence classes. Let the action $V \times S \to V$ be such that, for all C, s, one has $d(C, Cs) = 1$ and no $(n-1)$-codimensional cell in s is contained in $C \cap Cs$. Let $m(C; s, t)$ be half the number of chambers containing $C \cap Cs \cap Ct$.

A *realisation* of a coloured graph Γ is a gsa $\mathcal{A}$ such that $G(\mathcal{A}) \cong \Gamma$. For a fully coloured graph, a realisation may not exist, and it may not be unique up to isomorphism if it exists.

A celebrated result by Tits [B, section 5.4.4], [V], [H, section 5.13] states that every Coxeter group (seen as a fully coloured graph) can be realised. Moreover, the realisation can be chosen to be covariant under some W-action, so that we have a faithful linear representation $W \to \mathrm{GL}(Q)$.

It is not hard to generalise Tits's result, with essentially the same proof, to theorem 15 which roughly states that something is a gsa as soon as its 3-residues are, or even if certain small parts of 3-residues are. The challenge lies in finding interesting examples where theorem 15 can be used to prove that something is a gsa. To arrive at such examples, one has to overcome two obstacles which are easy in the case of Coxeter groups: (a) to find a fully coloured graph; and (b) to find a realisation of it.

We give a modest partial solution to part (b) as follows. Define a $(2,3)$-graph to be a fully coloured graph (V, S, m) such that $m(v; s, t) \in \{2, 3\}$ for all v, s, t and such that all 3-residues are either of the common Coxeter type, or of type $A(3, 7)$ defined in figure 6(b). (In the main text we use a different but equivalent definition of $(2,3)$-graphs.) Then every $(2,3)$-graph is realisable. Without much more effort one proves a more general result involving so-called $(2, 3, \infty)$-graphs which we also include (see theorem 24 and proposition 32).

The fully coloured graph associated with a Coxeter system (W, S) is a $(2,3)$-graph if and only if the Coxeter system is simply laced, that is, the order of st is in $\{1, 2, 3\}$ for all $s, t \in S$.

For $n \geq 0$, let K_n be the group presented by a set $T_n \subset K_n$ of $\binom{n+1}{2}$

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[1] A hyperplane is a 1-codimensional linear subspace.

[2] Locally finite in U^0 means that every compact subset of U^0 meets finitely hyperplanes from $\mathcal{A}$.

[3] A chamber is the closure of a connected component of $U^0 \setminus (\cup\mathcal{A})$.

[4] A simplicial chamber is one of the form $\{x \in Q \mid f_i(x) \geq 0 \text{ for all } i\}$ for some independent set of linear maps $\{f_i \mid i \in I\}$.

generators written

$$T_n = \left\{ t(a,b) = \binom{a}{b} \;\middle|\; a, b \in \{0, 1, \ldots, n\},\ a < b \right\}$$

and relations s^2 for all $s \in T_n$ and

$$\binom{a}{b}\binom{c}{d}\binom{a}{b}\binom{c}{d}$$

whenever $0 \le a < b \le c < d \le n$;

$$\binom{a}{b}\binom{a+x}{b-y}\binom{a}{b}\binom{a+y}{b-x}$$

whenever $x, y \ge 0$ and $0 \le a < a + x + y < b \le n$; and

$$\binom{a}{b-z}\binom{a+y}{b}\binom{a}{b-x}\binom{a+z}{b}\binom{a}{b-y}\binom{a+x}{b}$$

whenever $x, y, z > 0$ and $0 \le a \le a + x + y + z = b \le n$.

This construction is motivated by the observation that there exists a K_n-action on $\{1, \ldots, n\}$ given by

$$\binom{a}{b}(x) = \begin{cases} a + b + 1 - x & \text{if } a + 1 \le x \le b, \\ x & \text{otherwise.} \end{cases}$$

Our main result (theorem 49) states that there exists a K_n-action on a $(2,3)$-graph $\Gamma_n = (V, S, m)$ such that the action on V is simply transitive, and such that there exists a vertex $v \in V$ such that, for all $g \in K_n$, we have $d(v, gv) = 1$ if and only if $g \in T_n$. We give a case-by-case proof of the theorem by looking at every 3-residue separately.

As we remarked above, $(2,3)$-graphs are realisable. In particular, K_n has a geometric representation much as Coxeter groups have.

Residues of realisable fully coloured graphs (seen as fully coloured graphs themselves) are again realisable. A residue in a Coxeter fully coloured graph is again Coxeter. Contrary to this, a residue in Γ_n is not necessarily isomorphic to any Γ_k. We call such residues *admissible graphs* and we study them on a par with Γ_n itself.

Among the $(2,3)$-graphs the finite ones seem most interesting. We list the irreducible rank 4 $(2,3)$-graphs without proof in proposition 52. Two of them are Coxeter and two of them are not. Both of the non-Coxeter ones are admissible. This suggests that Γ_n may be a good source for finite $(2,3)$-graphs.

Section 2 introduces coloured graphs and proves Tits's result in our wider setting. This also generalises the fully coloured graphs mentioned above. In section 3 we define $(2,3,\infty)$-graphs (which in the main text are by definition realisable) and classify them locally. In section 4 we study the group K_n and its relation with $(2,3)$-graphs.

2 Realisations of coloured graphs

A *partial map* $f\colon A \to B$ (of sets, say) consists of a subset $D \subset A$ and a map $D \to B$. We call D the domain of f and we say that $f(a)$ is not defined unless $a \in D$. A statement such as "$f(a)$ is positive" implies in particular that $f(a)$ is defined.

For a set S, let F_S be the free monoid on S. We consider S to be a subset of F_S. If $S \subset T$ then $F_S \subset F_T$.

Definition 1. A *coloured graph* is a tuple (V, S, A, m) with the following properties. Firstly, V is a set (of vertices) and S is a set (of colours). We have a partial action $A\colon V \times F_S \to V$ written $(v, g) \mapsto vg$, that is, a partial map such that for all $v \in V$, $g, h \in F_S$, if $v(gh)$ or $(vg)h$ is defined then so is the other, and they are equal. Usually we omit A from the notation. We have a partial map $m\colon V \times S \times S \to \mathbb{Z}_{\geq 1} \cup \infty$ such that $m(v; s, t)$ is defined if and only if, for all $k > 0$, the vertices $v(st)^k$ and $v(ts)^k$ are defined. Moreover, the following hold.

- If vs is defined ($v \in V$, $s \in S$) then $(vs)s = v$.
- We have $m(v; s, t) = 1$ if and only if $s = t$ and vs is defined. (Recall that if vs is not defined then neither is $m(v; s, t)$.)
- Suppose that $m(v; s, t)$ is defined. Then $m(v; s, t) = m(v; t, s)$ and $m(v; s, t) = m(vs; s, t)$. (2)
- If $k := m(v; s, t)$ is defined and finite then $v(st)^k = v$. (3)
- The set V is universal. That is, let (V', S, m') satisfy the above too and let $f\colon V' \to V$ be a map satisfying (i) $(fv)s = f(vs)$ for all $(v, s) \in V' \times S$ such that at least one side is defined; (ii) whenever $m'(v; s, t)$ is defined, it equals $m(fv; s, t)$. Then the restriction of f to every F_S-orbit in V' is injective. (4)

A *fully coloured graph* is a coloured graph (V, S, m) such that the action $V \times S \to V$ (hence m) is everywhere defined. A direct definition of fully coloured graphs was given in the introduction. □

Let (V, S, m) be a coloured graph and let $I \subset S$. An *I-residue* is a subset of V of the form $\{vg \mid g \in F_I\}$ where $v \in V$ (of course, only the defined vertices are included). We call it also an *r-residue* if $r = \#I$.

Let R be the $\{s, t\}$-residue through v. It follows from (2) that $m(v; s, t)$ depends only on (R, s, t) (if it is defined). We write it as $m(R; s, t)$ accordingly.

As explained in the introduction, every Coxeter system gives rise to a fully coloured graph.

Remark 5. Let (V, S, m) be a coloured graph.

(a). There is an equivalence relation on V with two equivalence classes such that v, vs are not equivalent for all $v \in V$, $s \in S$. In particular, $v \neq vs$. This follows from the universality (4) and the fact that the relations (3) have even length.

(b). Let R be an $\{s,t\}$-residue. Then $\#R$ divides $2m(R;s,t)$, but it may happen that $\#R \neq 2m(R;s,t)$, as is shown by the following example. Put $V=(\mathbb{Z}/2)^3$ and $S=\{r,s,t\}\subset V$ where $r=(1,0,0)$, $s=(0,1,0)$, $t=(0,0,1)$. Let S act on V by right multiplication. Define $m(v;a,b)=2$ for all v,a,b except if $\{a,b\}=\{s,t\}$ and $v\in R:=\langle s,t\rangle$ in which case we put $m(v;s,t)=4$. Then (V,S,m) is a coloured graph but $2m(R;s,t)=8\neq 4=\#R$.

(c). From section 3 we shall only consider *fully* coloured graphs. We take the opportunity to prove the results of this section in the slightly more general setting of partial maps.

Let Q be a real vector space. A *hyperplane* in Q is a 1-codimensional linear subspace. An *open (respectively, closed) half-space* is a subset of Q of the form $f^{-1}(\mathbb{R}_{>0})$ (respectively, $f^{-1}(\mathbb{R}_{\geq 0})$) where $f\colon Q\to\mathbb{R}$ is a nonzero linear map. If H is one of the above half-spaces, then the *boundary* ∂H is defined to be $f^{-1}(0)$.

We call a coloured graph (V,S,m) *connected* if for all $v,w\in V$, there exists $g\in F_S$ such that $vg=w$.

Definition 6. Let $\Gamma=(V,S,m)$ be a connected coloured graph. A *realisation* of Γ consists of the data (7)–(8) satisfying properties (9)–(11) below.

- For every $v\in V$ a real vector space $P(v)$ with basis $\{p(v,s)\mid s\in S\}$ (a set in bijection with S). (7)
- Whenever $w:=vs$ is defined ($v\in V$, $s\in S$) an isomorphism (8)

$$\phi_{v,s}\colon P(v)\to P(w)$$

  such that $p(v,t)\,\phi_{vs}=p(w,t)$ for all $t\in S\smallsetminus s$.
- Let Q denote the quotient of the disjoint union $\sqcup_{v\in V}P(v)$ by the smallest equivalence relation $\equiv$ such that $x\,\phi_{vs}\equiv x$ for all v,s and all $x\in P(v)$. Then the natural map $P(v)\to Q$ is bijective for one hence all $v\in V$. (9)

Note that the condition (9) is equivalent to $\phi_{v_1s_1}\cdots\phi_{v_ns_n}=1$ (indices in $\mathbb{Z}/n$) whenever $v_is_i=v_{i+1}$ for all i. It is sufficient for this to hold for $\#\{s_1,\ldots,s_n\}=2$, by (4).

The image in Q of $p(v,s)$ is written $q(v,s)$. It follows from (9) that Q is a real vector space with basis $\{q(v,s)\mid s\in S\}$ (a set in bijection with S) whenever $v\in V$. For $v\in V$ we define the *chamber* $C(v)=\sum_{s\in S}\mathbb{R}_{\geq 0}\,q(v,s)$.

- We have $C(v)^0\cap C(vs)^0=\varnothing$ for all $v\in V$, $s\in S$, where 0 denotes the relative interior. (10)
- Let $R\subset V$ be an $\{s,t\}$-residue, $s\neq t$. Suppose that $X=\cap_{v\in R}C(v)$ has codimension 2, that is, $\#R\geq 2$. (11)

  If $k=m(R;s,t)$ is defined and finite then there exist k (distinct) hyperplanes in Q containing X such that every component of the complement of these hyperplanes meets $C(v)$ for a unique $v\in R$. In particular, $\#R=2m(R;s,t)$.

If $m(R; s, t)$ is infinite or not defined then $\cup_{v \in R} C(v)$ is contained in some closed half-space whose boundary contains X. $\square$

Suppose $vs = w$ ($v \in V$, $s \in S$). Then there are unique $c_t \in \mathbb{R}$ ($t \in S$) such that

$$q(w, s) = \sum_{t \in S} c_t \, q(v, t).$$

Now (10) is equivalent to $c_s < 0$.

Example 12. It is not hard to show that every Coxeter coloured graph admits a (covariant) realisation determined by

$$p(v, s) \, \phi_{v,s} = -p(vs, s) + \sum_{t \in S \smallsetminus \{s\}} 2 \cos \frac{\pi}{m(s, t)} \, p(vs, t). \tag{13}$$

See [B, section 5.4.3], [H, section 5.3], [V].

Remark 14. Suppose that the coloured graph (V, S, m) admits a realisation. Let $v \in V$ and let $s, t \in S$ be distinct. If $m(v; s, t)$ is defined (but possibly infinite) then the $\{s, t\}$-residue through v has $2m(v; s, t)$ elements. This follows immediately from (11). In particular, $vs \neq vt$.

In the case of Coxeter groups, this is the usual proof that the order of st equals $m(s, t)$ rather than a proper divisor of it.

Let (V, S, m) be a connected coloured graph. For $v, w \in V$, define $d(v, w)$ to be the least $k \geq 0$ such that there are $s_1, \ldots, s_k \in S$ with $vs_1 \cdots s_k = w$. Then d is a metric. By a *semi-geodesic* we mean a tuple $(v_1, \ldots, v_n)$ of vertices such that $d(v_1, v_n) = \sum_{i=1}^{n-1} d(v_i, v_{i+1})$.

For $v \in V$, $s \in S$, we define

$$H(v, s) := \Big\{ \sum_{t \in S} c_t \, q(v, t) \;\Big|\; c_t \in \mathbb{R} \text{ for all } t \in S \text{ and } c_s \geq 0 \Big\} \subset Q.$$

If vs exists then $H(v, s)$ is the closed half-space in Q containing $C(v)$ whose boundary contains $C(v) \cap C(vs)$.

In the remainder of this section, we consider a connected coloured graph with a realisation, and use the above notation.

The proof of the following is similar to [B, section 5.4.4].

Theorem 15. *Let $v', w \in V$ be distinct. Let $s \in S$ and suppose that either (a) $v's$ is not defined, or (b) $v := v's$ is defined and (v, v', w) is a semi-geodesic. Then $C(w) \subset H(v', s)$.*

Proof. Induction on $n = d(v', w)$. For $n = 0$ it is trivial. If $n \geq 1$, let $v'' = v't$ ($t \in S$) be a neighbour of v' such that (v', v'', w) is a semi-geodesic. Note that $s \neq t$. In case (b), we have $v'' \neq v$.

Let R be the $\{s, t\}$-residue through v'. For $a, b \in R$, let $d_0(a, b)$ be the least $k \geq 0$ such that there exist $s_1, \ldots, s_k \in \{s, t\}$ with $b = as_1 \cdots s_k$. So $d_0(a, b) \geq d(a, b)$.

Let A denote the set of those $a \in R$ for which $d(v', w) = d_0(v', a) + d(a, w)$. Let $x \in A$ be an element with $d(x, w)$ minimal.

We have $\#R \geq 2$ because $v', v'' \in R$. Let $y \in R$ be a neighbour of x, that is, $d_0(x, y) = 1$.

We claim that (y, x, w) is a semi-geodesic. If not, we would have $d(w, y) = d(w, x) - 1$ and hence

$$\begin{aligned} d(w, v') &\leq d(w, y) + d(y, v') \leq d(w, y) + d_0(y, v') \\ &= (d(w, x) - 1) + d_0(y, v') \\ &\leq d(w, x) - 1 + d_0(x, v') + 1 = d(w, v'). \end{aligned}$$

So equality holds throughout, forcing $d(w, v') = d(w, y) + d_0(y, v')$, and therefore $y \in A$, contrary to $d(w, y) < d(w, x)$.

Note that $v'' \in A$, whence $d(w, x) \leq d(w, v'') < d(w, v')$. Therefore we may apply the induction hypothesis to the triples (x, w, r) for $r \in \{s, t\}$. We find that

$$C(w) \subset H(x, s) \cap H(x, t). \tag{16}$$

In case (a) we have $H(x, s) \cap H(x, t) \subset H(v', s)$ so by (16) we find $C(w) \subset H(v', s)$ as required. Suppose now that we're in case (b). Then $d_0(x, v) > d_0(x, v')$, since otherwise

$$\begin{aligned} d(w, v) &\leq d(w, x) + d(x, v) \leq d(w, x) + d_0(x, v) \\ &< d(w, x) + d_0(x, v') = d(w, v'), \end{aligned}$$

a contradiction. By (4), this shows that $H(x, s) \cap H(x, t) \subset H(v', s)$ and, on combining with (16) as before, $C(w) \subset H(v', s)$. □

Corollary 17. *If $v, w \in V$ are distinct then $C(v)^0 \cap C(w)^0 = \varnothing$.*

Proof. Let (v, v', w) be a semi-geodesic with $v' = vs$, $s \in S$. Apply theorem 15. □

A *cell* is a set of the form $\sum_{s \in I} \mathbb{R}_{\geq 0}\, q(v, s)$ (which is $\{0\}$ if $I = \varnothing$) for $v \in V$, $I \subset S$.

Corollary 18. *Let X, Y be distinct cells. Then $X^0 \cap Y^0 = \emptyset$.*

Proof. Let v, w be vertices such that $X \subset C(v)$, $Y \subset C(w)$ with $n = d(v, w)$ minimal. (We don't assume that X is a "face" of $C(v)$ or Y is of $C(w)$.) If $n = 0$ it is trivial so suppose $n > 0$. Let $v' = vs$ be a neighbour of v such that (v, v', w) is a semi-geodesic. Then $X \not\subset C(v')$ by minimality of n. So $X^0 \cap H(v', s) = \varnothing$. We also have $Y \subset C(w) \subset H(v', s)$ so $X^0 \cap Y^0 = \varnothing$. □

The union of all $C(v)$ is denoted U.

Corollary 19. *The following hold.*

(a). *U is convex.*

(b). *For all $x, y \in U$, the line segment $[x, y] := \{tx + (1 - t)y \mid 0 \leq t \leq 1\}$ meets finitely many cells of U.*

Proof. By corollary 18 we can prove parts (a) and (b) at once by showing that for all $x, y \in U$, the line segment $[x, y]$ is contained in the union of finitely many cells. Let v, w be vertices with $x \in C(v)$, $y \in C(w)$, $n = d(v, w)$ minimal. Induction on n. If $n = 0$ it is trivial. If $n > 0$, write $[x, y] \cap C(v) = [x, z]$. Since $y \notin C(v)$, we have $y \notin H(v, s)$ for some $s \in S$ with $z \in \partial H(v, s)$. Since $y \in C(w) \backslash H(v, s)$, it follows from theorem 15 that (i) $v' := vs$ is defined, and (ii) $d(v', w) < d(v, w)$. Since $z \in C(v')$, the segment $[z, y]$ is contained in finitely many cells by induction. Moreover, $[x, z]$ is clearly contained in finitely many cells. This proves the induction step which finishes the proof. □

3 $(2, 3, \infty)$-Graphs

From now on, all our coloured graphs are fully coloured.

Definition 20. A $(2, 3, \infty)$-*graph* is a connected fully coloured graph (V, S, m) which admits a (necessarily essentially unique) realisation (7)–(11) with the following properties.

- We have $m(v; s, t) \in \{2, 3, \infty\}$ for all v, s, t. (21)
- We define a bijection $N\colon \{2, 3, \infty\} \to \{0, 1, 2\}$ by $N(2) = 0$, $N(3) = 1$, $N(\infty) = 2$. Equivalently, $N(k) = 2\cos(\pi/k)$. We put $n(v; s, t) := N\big(m(v; s, t)\big)$ and $n(R; s, t) = n(v; s, t)$ if R is the $\{s, t\}$-residue through v. (22)

  Suppose $vs = w$ ($v \in V$, $s \in S$). Then

$$\begin{aligned} p(v, s)\, \phi_{vs} &= -p(w, s) + \sum_{t \in S \smallsetminus \{s\}} n(v; s, t)\, p(w, t) \\ &= -p(w, s) + \sum_{t \in S \smallsetminus \{s\}} \cos \frac{\pi}{m(v; s, t)}\, p(w, t) \end{aligned}$$

  Compare with (13).

The realisation with these properties is called the *standard realisation* in order to distinguish it from other realisations, if any. Note that the uniqueness of the standard realisation follows immediately from (22). □

Our next aim is to provide an explicit criterion for a coloured graph to be a $(2, 3, \infty)$-graph. We need the notion of structure sequence, which we shall now define (see figure 1).

Definition 23. Let (V, S, m) be a coloured graph satisfying (21). Let $v \in V$, let $s, t \in S$ be distinct, and suppose that $k := m(v; s, t)$ is defined and finite. Define v_i ($i \in \mathbb{Z}/2k$) by $v_0 = v$, $v_{2i-1}t = v_{2i} = v_{2i+1}s$ for all i (see figure 1). The map

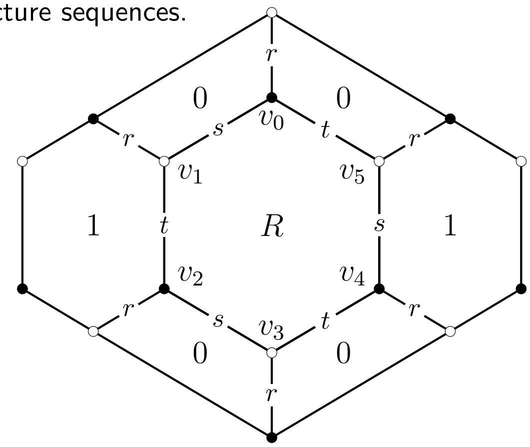
$$\begin{aligned} \mathbb{Z}/2k &\longrightarrow \{0, 1, 2\} \\ 2i &\longmapsto n(v_{2i}; r, s) \\ 2i+1 &\longmapsto n(v_{2i+1}; r, t) \end{aligned}$$

is called the *structure sequence* of the $\{s,t\}$-residue R through v. We always consider two structure sequences to be equal if they differ only by a cyclic permutation or reversal. Therefore the structure sequence is determined by (R,s,t).

If $m(R;s,t)$ is infinite or undefined then we don't consider an associated structure sequence. $\square$

Figure 1. Structure sequences.

This picture shows part of a 3-residue T containing an $\{s,t\}$-residue $R=\{v_i \mid i\}$ with $m(v_0;s,t)=3$. In the middle of every 2-residue R_i in T meeting R in an edge $\{v_i,v_{i+1}\}$ the picture shows the value of $n(R_i;s,t)$. The structure sequence for R is $(0,0,1,0,0,1)$.



Theorem 24. (a). *Let Γ be a connected fully coloured graph satisfying (21). Then Γ is a $(2,3,\infty)$-graph if and only if the following hold.*

- *All structure sequences of length 4 are of the form (n_1,n_2,n_1,n_2), $n_1,n_2\in\{0,1,2\}$.* (25)
- *All structure sequences of length 6 are of the form $(n_i)_{i\in\mathbb{Z}/6}$ where $n_i\in\{0,1,2\}$ and where $(-1)^i(n_i-n_{i+3})$ is independent on i.* (26)

(b). *The length 6 structure sequences satisfying the condition of* (26) *are precisely*

$(0,0,0,0,0,0)$	$(0,0,2,0,0,2)$	$(1,2,2,1,2,2)$
$(0,0,1,0,0,1)$	$(0,2,0,2,0,2)$	$(2,2,2,2,2,2)$
$(0,1,0,1,0,1)$	$(0,2,2,0,2,2)$	$(1,1,1,0,2,0)$
$(0,1,1,0,1,1)$	$(1,1,2,1,1,2)$	$(1,1,1,2,0,2)$
$(1,1,1,1,1,1)$	$(1,2,1,2,1,2)$	$(0,1,2,0,1,2)$

up to cyclic permutation and reversing.

Proof. Define $P(v)$ $(v\in V)$ and $\phi_{vs}\colon P(v)\to P(vs)$ uniquely by (7), (8), (22). By the definition of realisations of coloured graphs, Γ is a $(2,3,\infty)$-graph if and only if (9), (10) and (11) hold.

Let $P^*(v)$ be the dual to $P(v)$. Let $\langle\cdot,\cdot\rangle\colon P(v)\times P^*(v)\to\mathbb{R}$ be the natural pairing and let $\{p^*(v,s)\mid s\in S\}$ be the dual basis of $P^*(v)$ defined by

$$\big\langle p(v,s),p^*(v,t)\big\rangle=\begin{cases}1 & \text{if } s=t,\\ 0 & \text{otherwise.}\end{cases}$$

Then ϕ_{vs}^{-1} induces a map $\phi_{vs}^*\colon P^*(v) \to P^*(w)$. For all $v \in V$ and all distinct $s, t \in S$ we have

$$\begin{aligned} p^*(v,s)\,\phi_{vs} &= -p^*(vs,s), \\ p^*(v,t)\,\phi_{vs} &= p^*(vs,t) + n(v;s,t)\,p^*(vs,s). \end{aligned} \tag{27}$$

Let $(28_{k=2})$ and $(28_{k=3})$ denote the relevant special cases of the following statement.

- Let v_0, s, t be such that $m(v_0;s,t) = k$. Let $\{v_i \mid i \in \mathbb{Z}/2k\}$ be the $\{s,t\}$-residue through v_0, and $v_i s_i = v_{i+1}$ for all i, and $s_i = s$ for even i and $s_i = t$ for odd i. Then $\phi_{v_1 s_1} \cdots \phi_{v_n s_n} = 1$. (28)

Then (9) is equivalent to $(28_{k=2})$ and $(28_{k=3})$. We begin by proving that $(28_{k=3})$ is equivalent to (26) if $\#S \geq 3$. Let s, t, v_i, s_i be as in $(28_{k=3})$ and let $r \in S \smallsetminus \{s,t\}$. Define the rows of vectors

$$f_i := \big(p^*(v_i,s), p^*(v_i,t), p^*(v_i,r)\big) \qquad \text{if } i \text{ is even,} \tag{29}$$

$$f_i := \big(p^*(v_i,t), p^*(v_i,s), p^*(v_i,r)\big) \qquad \text{if } i \text{ is odd.} \tag{30}$$

After interchanging s, t if necessary, we have $f_i\,\phi_{v_i s_i} = f_i\,M(n_i)$ for all i, where $\phi_{v_i s_i}$ acts componentwise and

$$M(n) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & n & 1 \end{pmatrix}.$$

We have

$$M(n_1)\,M(n_2)\,M(n_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & n_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & n_2 & 1 \end{pmatrix} M(n_3)$$

$$= \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ n_1 & n_1+n_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & n_3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ n_1+n_2 & n_2+n_3 & 1 \end{pmatrix} \tag{31}$$

which is an involution. It follows that

$$\begin{aligned} \prod_{i=1}^{6} M(n_i) = 1 &\iff M(n_1)\,M(n_2)\,M(n_3) = M(n_4)\,M(n_5)\,M(n_6) \\ &\iff (n_4,n_5,n_6) = (n_1,n_2,n_3) + (k,-k,k) \text{ for some } k \\ &\iff (-1)^i(n_i - n_{i+3}) \text{ is independent of } i. \end{aligned}$$

We have proved that $(28_{k=3})$ is equivalent to (26) if $\#S \geq 3$. In case $\#S < 3$ the proof is the same as above except that r is absent, that is, the last row and column of $M(n)$ are removed.

Next we prove that $(28_{k=2})$ is equivalent to (25). Let s, t, v_i, s_i be as in $(28_{k=2})$ and let $r \in S \smallsetminus \{s,t\}$. As before, define f_i by (29), (30). After interchanging s, t if necessary, we have $f_i\,\phi_{v_i s_i} = f_i\,L(n_i)$ for all i where

$$L(n) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & n & 1 \end{pmatrix}.$$

Now

$$L(n_1)\,L(n_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & n_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & n_2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ n_1 & n_2 & 1 \end{pmatrix}$$

from which it readily follows that

$$L(n_1)\cdots L(n_4) = 1 \iff (n_1, n_2, n_3, n_4) = (n_1, n_2, n_1, n_2).$$

This proves that $(28_{k=2})$ is equivalent to (25). (The case $\#S < 3$ is again a consequence of the same computation). Therefore, (9) is equivalent to (25) and (26). Assume now (9). It remains to prove (10) and (11).

Condition (10) states that neighbouring open chambers are disjoint; it holds because of the negative sign in (27).

Finally, we prove (11) in the case where $m(R; s, t) = 3$, leaving the other cases to the reader. On writing $W := \sum_{u \in S \smallsetminus \{s,t\}} \mathbb{R}\, p(v_1, u)$ and $g_i := (p(v_i, s) + W, p(v_i, t) + W)$ we need to prove $g_i + g_{i+3} = 0$. Well, we have $g_{i+3} = g_i K$ where K is the transpose of $M(n_1)\,M(n_2)\,M(n_3)$ (which we computed in (31)) without the last row and last column. This proves the promised case of (11).

This finishes the proof of (a). Part (b) is straightforward. □

The *rank* of a coloured graph (V, S, m) is defined to be $\#S$. By a $(2,3)$-*graph* we mean a $(2, 3, \infty)$-graph (V, S, m) such that $m(v; s, t) \in \{2, 3\}$ for all v, s, t. We aim to classify the $(2, 3)$-graphs of rank 3.

The *product* of two coloured graphs (V_i, S_i, m_i) $(i = 1, 2)$ is defined to be $(V_1 \times V_2, S_1 \sqcup S_2, m)$ ($\sqcup$ is disjoint union) where

$$\begin{aligned} (v_1, v_2)\, s_1 &= (v_1 s_1, v_2) && \text{if } v_i \in V_i \text{ for all } i \text{ and } s_1 \in S_1, \\ (v_1, v_2)\, s_2 &= (v_1, v_2 s_2) && \text{if } v_i \in V_i \text{ for all } i \text{ and } s_2 \in S_2, \\ m\big((v_1, v_2); s, t\big) &= m_i(v_i; s, t) && \text{if } s, t \in S_i, \\ m\big((v_1, v_2); s_1, s_2\big) &= 2 && \text{if } s_i \in S_i \text{ for all } i. \end{aligned}$$

A coloured graph is *irreducible* if it is not isomorphic to a product of two coloured graphs of positive rank. It is clear that the product of two $(2, 3, \infty)$-graphs is again a $(2, 3, \infty)$-graph.

Proposition 32. *Up to isomorphism there are just three irreducible $(2,3)$-graphs of rank 3:*

$$A_3,\ \tilde{A}_2,\ A(3, 7).$$

Here, A_3, $\tilde{A}_2$ are the usual names[5] *of Coxeter groups while $A(3, 7)$ is defined*[6] *in figure 6(b) (more precisely, it is the $(2, 3)$-graph dual to the figure) and which can also be defined by the hyperplane arrangement*

$$xyz(x + y)(y + z)(z + x)(x + y + z) = 0.$$

Moreover, $\tilde{A}_2$ is infinite while the other two are finite.

[5] For names of Coxeter groups, see [B, section 6.4.1], [H, 2.4].

[6] In [G1] this arrangement is called $A_1(7)$ and in [G2] it is $A(7, 1)$.

Proof. Using theorem 24 this is an easy exercise involving drawings of graphs, and is left to the reader. $\square$

4 An example

4.1 An extension of the symmetric group

From now on we fix an integer $n \geq 0$. Let G_n be the free monoid on a set $T_n \subset G_n$ of $\binom{n+1}{2}$ elements written

$$T_n = \left\{ t(a,b) = \binom{a}{b} \;\middle|\; a, b \in \{0, 1, \ldots, n\},\ a < b \right\}.$$

A subset $R \subset G_n$ is *closed under cyclic permutations* if for all $a, b \in G_n$, if $ab \in R$ then $ba \in R$. (We call ba a *cyclic permutation* of ab).

We define $Q_n \subset G_n$ to be the smallest subset, closed under cyclic permutations, containing

$$\binom{a}{b}\binom{c}{d}\binom{a}{b}\binom{c}{d}$$

whenever $0 \leq a < b \leq c < d \leq n$;

$$\binom{a}{b}\binom{a+x}{b-y}\binom{a}{b}\binom{a+y}{b-x}$$

whenever $x, y \geq 0$ and $0 \leq a < a+x+y < b \leq n$; and

$$\binom{a}{b-z}\binom{a+y}{b}\binom{a}{b-x}\binom{a+z}{b}\binom{a}{b-y}\binom{a+x}{b}$$

whenever $x, y, z > 0$ and $0 \leq a \leq a+x+y+z = b \leq n$.

In order to motivate the definition of Q_n, note that the action of G_n on $\{1, \ldots, n\}$ defined by

$$\binom{a}{b}(x) = \begin{cases} a+b+1-x & \text{if } a+1 \leq x \leq b, \\ x & \text{otherwise} \end{cases}$$

has the property that the elements of Q_n act trivially.

Let K_n be the group presented by the generating set T_n and relations $s^2 = 1$ for all $s \in T_n$ and the relations in Q_n. One of our aims is to show that K_n is naturally the vertex set of a $(2,3)$-graph.

4.2 Admissible graphs

We observe now:

- For all distinct $a, b \in T_n$ there are unique $k \in \{2, 3\}$ and $c_3, \ldots, c_{2k} \in T_n$ such that $ab(c_3 \cdots c_{2k}) \in Q_n$. Also, $a^2 G_n \cap Q_n = \varnothing$ for all $a \in T_n$. (33)

- The set Q_n is also invariant under reversal, that is, under the anti-automorphism of G_n which fixes every element of T_n. (34)

Definition 35. We define an action $T_n \times G_n \to T_n$ written $(a, b) \mapsto a * b$ as follows. Firstly, $a * a = a$ for all $a \in T_n$. Let $a, b, c \in T_n$ and assume that Q_n meets $abcG_n$. Then $a * b = c$.

Note that this is well-defined by (33). Also note that $(a * b) * b = a$ for all $a, b \in T_n$ by (34).

Definition 36. For any set I, we define U_I to be the set of injective maps $I \to T_n$. Recall that F_I is the free monoid on I. We define an action $U_I \times F_I \to U_I$ written $(u, g) \mapsto u \triangledown g$ as follows. Let $u \in U_I$, $s \in I$.

- We put $[u \triangledown s](s) = u(s)$.
- Let $t \in I \smallsetminus \{s\}$ and put $a := u(t)$, $b := [u \triangledown t](s)$. Then $[u \triangledown ts](t)$ is $a * b$, that is, the unique $c \in T_n$ such that $abcG_n \cap Q_n \neq \varnothing$. □

Recall that a groupoid is a category all of whose morphisms are isomorphisms. If X, Y are objects of a category C we write $C(X, Y)$ for the set of morphisms of C from X to Y. All our categories are on the right, that is, the composition $C(X, Y) \times C(Y, Z) \to C(X, Z)$ is written $(f, g) \mapsto fg$ (rather than gf).

Definition 37. For any set I, we define the groupoid R_I with object set U_I by the presentation with generators

$$\begin{pmatrix} u \\ s \\ u \triangledown s \end{pmatrix} \in R(u, u \triangledown s) \qquad \text{whenever } u \in U_I,\ s \in I$$

and relations

$$\begin{pmatrix} u_0 \\ s \\ u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ t \\ u_2 \end{pmatrix} \begin{pmatrix} u_2 \\ s \\ u_3 \end{pmatrix} \begin{pmatrix} u_3 \\ t \\ u_4 \end{pmatrix} \cdots \begin{pmatrix} u_{2k-2} \\ s \\ u_{2k-1} \end{pmatrix} \begin{pmatrix} u_{2k-1} \\ t \\ u_{2k} \end{pmatrix}$$

whenever $u_0 = u_{2k}$ and either $s = t$ or

$$h := u_0(s)\, u_1(t)\, u_2(s)\, u_3(t) \cdots u_{2k-2}(s)\, u_{2k-1}(t) \tag{38}$$

is a power of an element of Q_n. □

For every (u_0, s, t) the possible values of k in the above are determined in the following easy result.

Lemma 39. *Let $u_0 \in U_I$, and let $s, t \in I$ be distinct. Define u_i $(i \in \mathbb{Z})$ by $u_{2i-1} \triangledown t = u_{2i} = u_{2i+1} \triangledown s$ for all i. Let g be the unique element of $u_0(s)\, u_1(t)\, G_n \cap Q_n$ and let $2p \in \{4, 6\}$ be its length. Let $q \in \mathbb{Z}_{\geq 0}$ and put $k = pq$. Define h by (38). Then $h = g^q$ and*

$$u_{2k}(r) = u_0(r) * g^q. \tag{40}$$

Therefore, $u_0 = u_{2k} \Leftrightarrow a * g^q = a$ for all $a \in u_0(I)$.

Proof. That $h = g^q$ follows readily from the definitions. For even i we have

$$\begin{aligned} u_i(r) * u_i(s) &= [u_i \triangledown r](r) * [u_i \triangledown rr](s) \\ &= [u_i \triangledown rrs](r) = [u_i \triangledown s](r) = u_{i+1}(r). \end{aligned}$$

Likewise, for odd i we have $u_i(r) * u_i(t) = u_{i+1}(r)$. By an obvious induction we find that $u_{2k}(r) = u_0(r) * h$ which proves (40). □

Definition 41. Let $u \in U_I$. We define a coloured graph $\Gamma(u) = (V, I, m)$ called an *admissible graph* as follows. Firstly, $V := R_I(u, -)$, the set of morphisms in R_I from u to any object. The action $V \times S \to V$ is defined by

$$(v, s) \mapsto vs := v \circ \begin{pmatrix} u_0 \\ s \\ u_0 \triangledown s \end{pmatrix} \qquad \text{whenever } v \in R_I(u, u_0).$$

We define m as follows. Use the notation of lemma 39 and let $v \in R_I(u, u_0)$. We define $m(v; s, t)$ to be the least $k > 0$ divisible by p such that $u_{2k} = u_0$. □

It is clear that $\Gamma(u)$ is a coloured graph. Notice that $\Gamma(u)$ has a natural base vertex $1_u \in R_I(u, u)$. Note that if u_1, u_2 are isomorphic objects of R_I then there is an isomorphism of coloured graphs $\Gamma(u_1) \to \Gamma(u_2)$ (preserving I pointwise) but it may not respect the base points.

Remark 42. Let $I \subset J$ be sets, $v \in U_J$ and $u := u|_I \in U_I$. It will follow from later results that there exists a unique injective map of coloured graphs (in the obvious sense) $\Gamma(u) \to \Gamma(v)$ preserving colours and base points. The injectivity is essentially a consequence of theorem 15. However, it is not even clear at this stage that such a map exists at all, because the values of m in $\Gamma(v)$ might be greater than those in $\Gamma(u)$. We shall find this unexpected aspect of admissible graphs helpful in the proof of theorem 49.

4.3 Equivalence relations

Recall that $T_n = \{t(a, b) \mid 0 \le a < b \le n\}$.

Definition 43. (a). For a subset $A \subset \{0, 1, \dots, n\}$ we define $T(A) := \{t(a, b) \mid a, b \in A,\ a < b\} \subset T_n$.

(b). Let $u \in U_I$. The *support* of u is defined to be $\operatorname{supp}(u) := \{a, b \mid t(a, b) \in u(I)\}$, that is, the smallest A such that $u(I) \subset T(A)$.

(c). Let $u_1, u_2 \in U_I$ and write $A_i = \operatorname{supp}(u_i)$ ($i \in \{1, 2\}$). We write $u_1 \sim u_2$ if there exists a map $f \colon A_1 \to A_2$ which is either an increasing bijection or a decreasing one, and $u_2 = g \circ u_1$ where $g \colon T(A_1) \to T(A_2)$ is defined by $g(t(a, b)) = t(fa, fb)$.

(d). For the sake of question 51, we include the following definition. Let $u_1, u_2 \in U_I$ and suppose $A = \operatorname{supp}(u_1) = \operatorname{supp}(u_2)$. By a cyclic permutation of A we mean a power of the permutation of A which takes every non-maximal element of A to the next bigger element of A. We say that u_1 is a *cyclic permutation* of u_2 if there exists a cyclic permutation f of A such that $u_2 = g \circ u_1$ where $g \colon T(A) \to T(A)$ is defined by $g(t(a, b)) = t(fa, fb)$. □

Clearly, $\sim$ is an equivalence relation on U_I.

Lemma 44. *Let $u_1, u_2 \in U_I$ be such that $u_1 \sim u_2$.*

(a). *Then $u_1 \triangledown g \sim u_2 \triangledown g$ for all $g \in F_I$.*

(b). *Write $E_j := u_j \triangledown F_I$. Then there is a unique isomorphism of F_I-sets $f\colon E_1 \to E_2$ (that is, a bijection such that $f(u \triangledown g) = (fu) \triangledown g$ for all $u \in E_1$, $g \in F_I$) such that $f(u_1) = f(u_2)$.*

(c). *For $j \in \{1, 2\}$, let $R_j \subset R_I$ be the component of u_j, that is, the biggest subcategory of R_I whose object set is E_j. Then there is a unique isomorphism of categories $g\colon R_1 \to R_2$ such that $g(u) = f(u)$ for all objects u (f as in (b)) and*

$$g\begin{pmatrix} u_3 \\ s \\ u_4 \end{pmatrix} = \begin{pmatrix} f(u_3) \\ s \\ f(u_4) \end{pmatrix}$$

whenever the left hand side has a meaning.

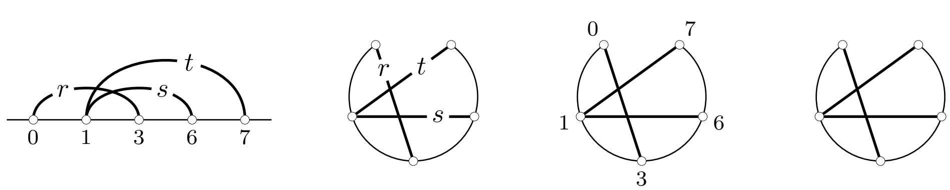
(d). *There is a unique isomorphism $\Gamma(u_1) \to \Gamma(u_2)$ of pointed coloured graphs which preserves I pointwise.*

Proof. Easy and left to the reader. □

Let $\approx$ be the equivalence relation on U_I generated by $\sim$ defined in definition 43 and $\cong$ (isomorphism in the groupoid R_I).

Let $\approx_s$ be the equivalence relation on U_I generated by $\approx$ and the graph of the symmetric group on I. In other words, $u_1 \approx_s u_2$ if and only if $u_1 \approx u_2 \circ s$ for some permutation s of I. Define $\sim_s$ and $\cong_s$ likewise.

It is natural to draw pictures of objects of R_I. The convention is easily understood from figure 2 which shows pictures for an element of U_I and some of its equivalence classes, and figure 3 which shows pictures for edges in admissible graphs.



(a) u (b) $u/{\sim}$ (c) $u/\mathrm{sym}(I)$ (d) $u/{\sim_s}$

Figure 2. Vertices of admissible graphs. Let $I = \{r, s, t\}$ have 3 elements. Part (a) shows a picture of the object $u \in U_I$ defined by $u(r) = t(0, 3)$, $u(s) = t(1, 6)$, $u(t) = t(1, 7)$. The picture in (a) is flat but we usually prefer the (equivalent) curled up version of (b)–(d). In (b) we see the $\sim$-class of u. The precise values $0, 1, 3, 6, 7$ are forgotten but their ordering is not as it is still shown in the picture. In (c) we divide out the symmetric group on I and in (d) we divide out $\sim_s$.

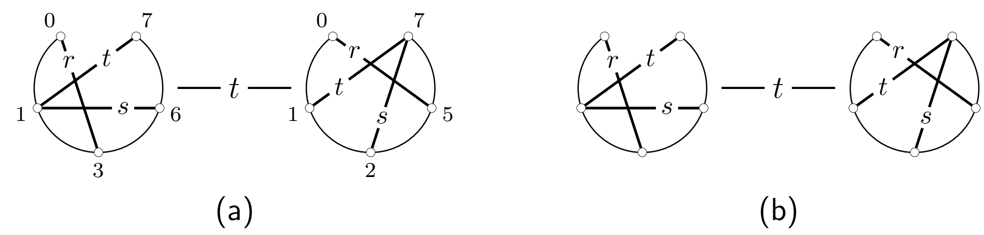


Figure 3. Edges of admissible graphs. Suppose that the left hand side in (a) depicts some $u \in U_I$ with $I = \{r, s, t\}$. Then the right hand side is $u \bigtriangledown t$. Part (b) is obtained from (a) by taking $\sim$-classes — we know by lemma 44 that coloured graphs survive division by $\sim$.

4.4 Main result

Example 45. We now have a detailed look at three rank 3 admissible graphs. Our understanding of them will be crucial in the case-by-case proof of theorem 49.

(a). One rank 3 admissible graph Γ is given in figure 5. You should verify it. Note that the vertices can be taken to be $\sim$-classes by lemma 44. The verification is helped by the schematic version of the graph in figure 4 and the order 6 automorphism group (which fixes the 2-residue numbered 1).

An observation which will be important in the proof of theorem 49 is that Γ is a $(2,3)$-graph. Indeed, it is isomorphic to the Coxeter graph of type A_3.

(b). Figure 6(a) shows part of another rank 3 admissible graph Γ. Convince yourself that it is correct. The dashed triangle is precisely 1/8 of the whole graph. The automorphism group of Γ is of order 8 and generated by the reflections in the edges of the dashed triangle.

Again, we observe that $\Gamma(u)$ is a $(2,3)$-graph (use theorem 24 or proposition 32). In the classification of rank 3 $(2,3)$-graphs (proposition 32) we said that it is of type $A(3,7)$. As every $(2,3)$-graph, it has a realisation as a hyperplane arrangement. This arrangement is shown in figure 6(b), which also serves to give a full picture rather than 1/8 of it.

(c). Let $u \in U_I$ be such that $u(I) = \{t(0,2), t(1,3), t(2,4)\}$. Then u is a single isomorphism class in R_I and one easily deduces that $\Gamma(u)$ must be a Coxeter coloured graph. Indeed it is of type A_3 and again it is a $(2,3)$-graph.

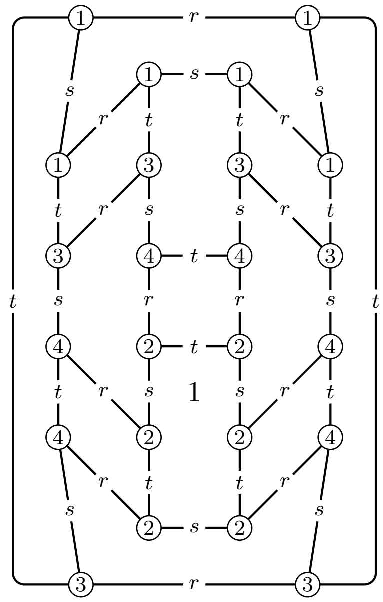
Definition 46. Let $u \in U_I$. We call u *reducible* if I can be written as the union of two non-empty disjoint sets A, B such that for all $(a,b) \in A \times B$ there exist $x, y \in T_n$ such that $u(a)\,u(b)\,x\,y \in Q_n$. Otherwise it is called *irreducible*.

Lemma 47. *Let $u \in U_I$. If u is reducible then $\Gamma(u)$ is reducible as a coloured graph.*

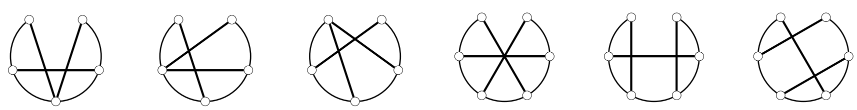
Proof. Left to the reader. $\square$

Lemma 48. *Suppose that $n \geq 4$ and I is a set of 3 elements.*

Figure 4. Schematic version of $\Gamma(u)$ for $u \in L_2 \cup L_3$. See figure 5 for a full picture. The automorphism group of this graph has order 6 and preserves the 2-residue labelled 1.



(a). *There are precisely six* $\sim_s$*-classes* $L_1, \ldots, L_6$ *of irreducible elements in* U_I*. They are given by the following representatives.*



L_1 L_2 L_3 L_4 L_5 L_6

(b). *The* $\approx_s$*-classes of irreducible elements in* U_I *are* L_1, $L_2 \cup L_3$ *and* $L_4 \cup L_5 \cup L_6$.

(c). *Every rank* 3 *admissible graph is a* $(2, 3)$*-graph.*

Proof. It is easy and left to the reader to prove (a) using lemma 47.

Proof of (b). The (connected) graph of example 45(a) and figure 5 involves L_2 and L_3 but no others (recall that reflection through a vertical line fixes every $\sim$-class by definition). Therefore $L_2 \cup L_3$ is a single $\approx_s$-class. Likewise, the graph of example 45(b) and figure 6(a) involves L_4, L_5 and L_6 but no others so $L_4 \cup L_5 \cup L_6$ is a $\sim_s$-class. Only one $\sim_s$-class L_1 remains which must therefore be a $\approx_s$-class as well; we looked at the related admissible graph in example 45(c).

Proof of (c). By (b) and lemma 47 we know all *irreducible* rank 3 admissible graphs. As we already observed in example 45, all of them are $(2, 3)$-graphs. It is easy and left to the reader to handle the reducible ones. $\square$

Theorem 49. *Every admissible graph is a* $(2, 3)$*-graph.*

Proof. Consider an admissible graph $\Gamma(u) = (V, I, m)$, $u \in U_I$.

Figure 5. The coloured graph $\Gamma(u)$ for $u \in L_2 \cup L_3$. A schematic version of it is shown in figure 4.

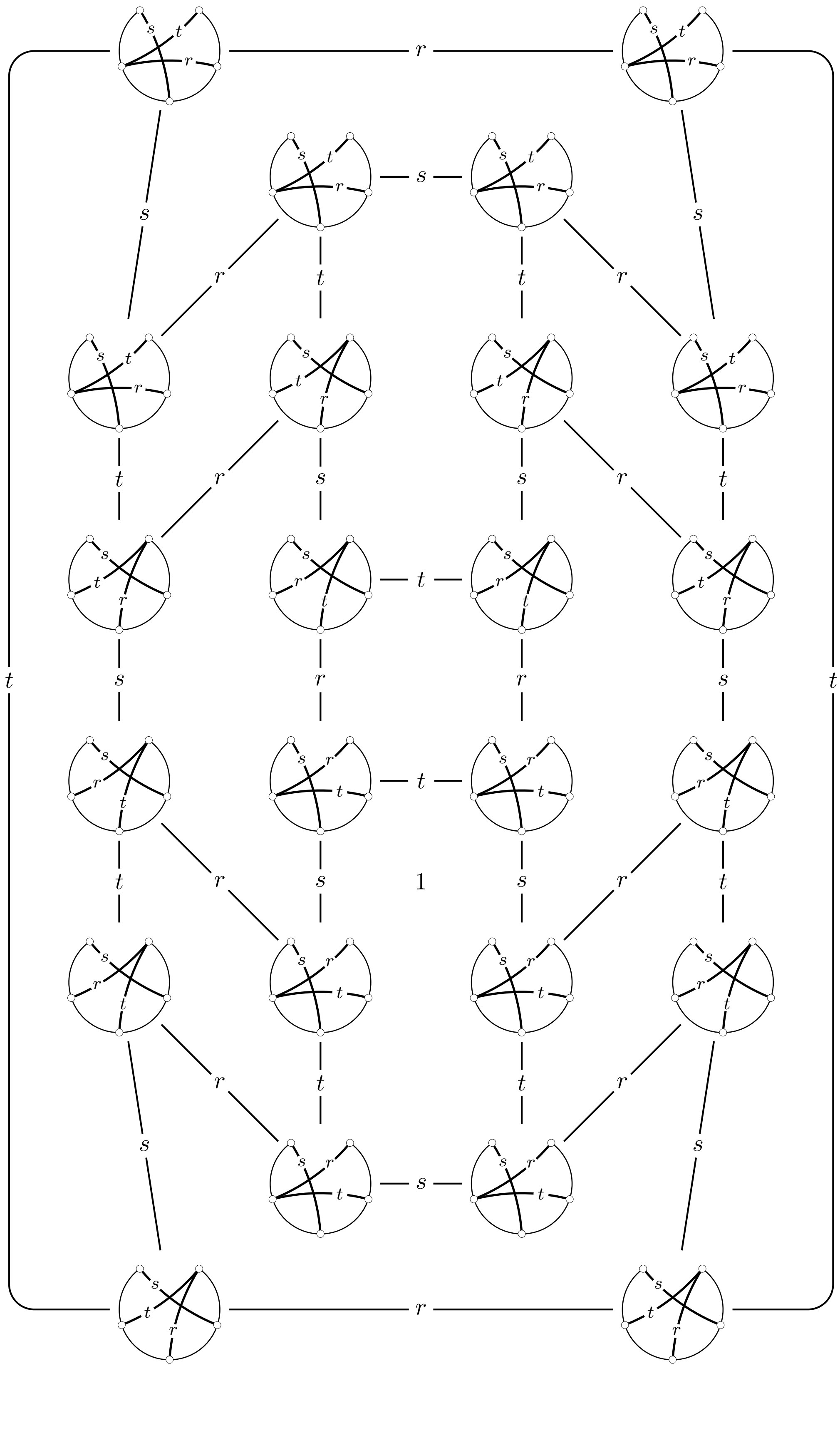
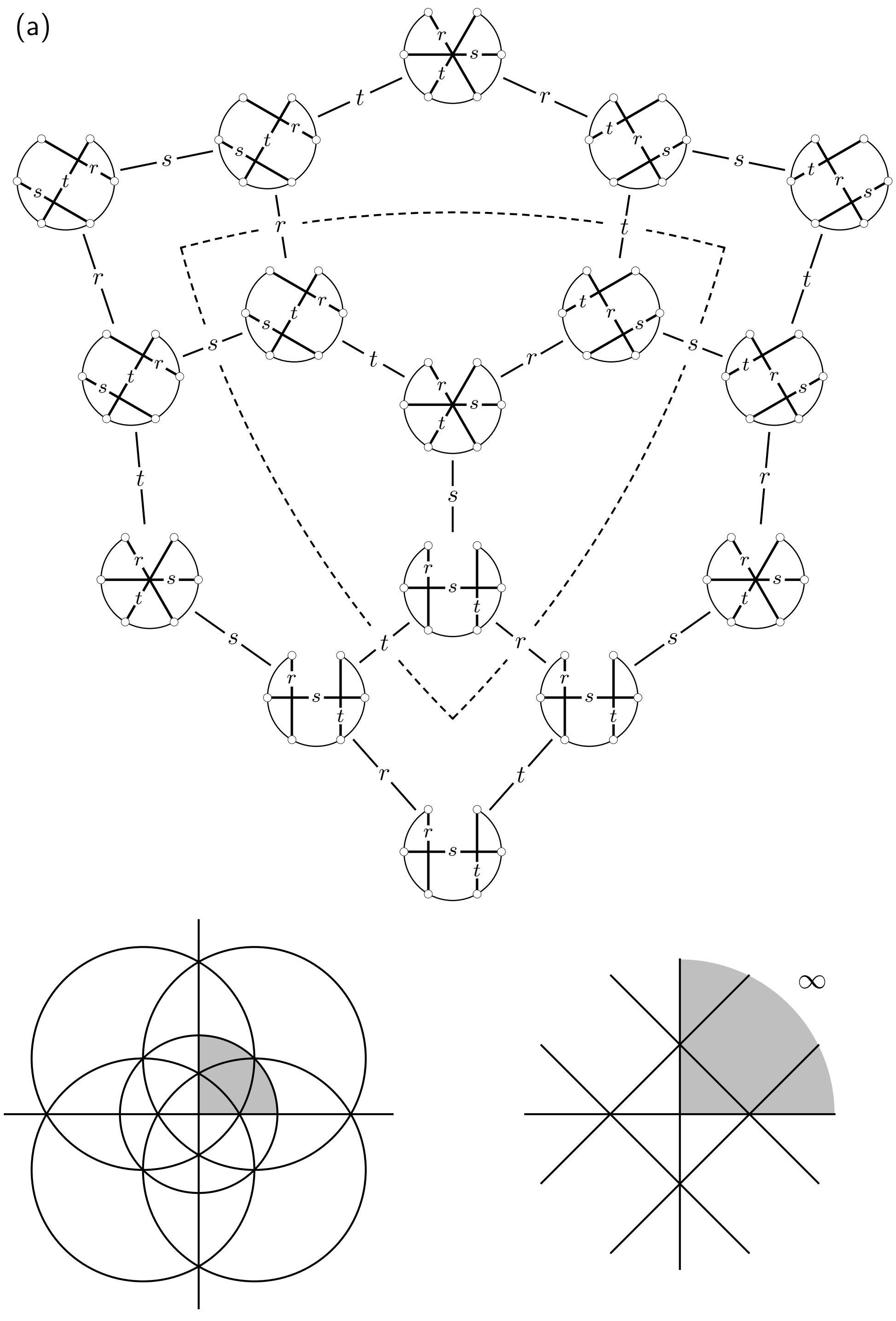


Figure 6. Picture (a). Part of the admissible graph $\Gamma(u)$ for $u \in L_4 \cup L_5 \cup L_6$. The dashed triangle is exactly one eighth of it and corresponds to the gray region of (b) and (c). **Pictures (b) and (c).** The line arrangement defined by $xyz(x+y)(y+z)(z+x)(x+y+z) = 0$. It is dual to the $(2,3)$-graph $A(3,7)$. The gray region is the dashed triangle of (a).



(b) Spherical picture of $A(3,7)$.

(c) Projective picture of $A(3,7)$. The line at infinity is included.

First we prove that $m(v;s,t) \in \{2,3\}$ for all v,s,t. In lemma 48 we observed this to be true in the rank 3 case. By lemma 39, this implies that $a * g = a$ for all $(a,g) \in T_n \times Q_n$. Using lemma 39 backwards we find that $m(v;s,t) \in \{2,3\}$ for all v,s,t.

Recall that a $(2,3)$-graph is just a $(2,3,\infty)$-graph for which $m(v;s,t)$ is never infinite. By theorem 24 it remains to prove that all structure sequences of $\Gamma(u)$ satisfy (25) and (26). But all structure sequences of all admissible graphs occur in rank 3 admissible graphs. In lemma 48 we already observed the latter to be $(2,3)$-graphs, in particular, to satisfy the required conditions (25) and (26). □

Corollary 50. *There exists a faithful linear representation of K_n.*

Proof. Let $u \in U_I$ be such that $u\colon I \to T_n$ is surjective. By theorem 49, $\Gamma(u)$ is a $(2,3)$-graph. It is clear that K_n acts on $\Gamma(u)$. By the unicity of the standard realisation of $(2,3)$-graphs, this action passes to a K_n-action on Q. The action on Q is faithful because the action on $\Gamma(u)$ is. □

Question 51. Recall that in definition 43(d) we defined cyclic permutations of elements of U_I. Observe now that every $u_1 \in L_1$ is a cyclic permutation of some $u_2 \in L_2$ (see lemma 48(a) for the classification of rank 3 admissible graphs). Also, $\Gamma(u_1)$ and $\Gamma(u_2)$ are isomorphic as coloured graphs because both are of Coxeter type A_3 as we saw in example 45(a) and (c). I don't know if this is a coincidence. Is it true in general that $\Gamma(u_3)$ and $\Gamma(u_4)$ are isomorphic whenever u_3 is a cyclic permutation of u_4?

We finish with a result without proof.

Proposition 52. *There are precisely four isomorphism classes of rank 4 irreducible finite $(2,3)$-graphs. They are the Coxeter ones A_4, D_4 and two more named $A(4,13)$, $A(4,15)$. Among them, D_4 is the only non-admissible one. Possible choices of u_{13}, $u_{15} \in U_I$ such that $A(4,13) = \Gamma(u_{13})$, $A(4,15) = \Gamma(u_{15})$ are as follows.*

$$u_{13}(I) = \{t(0,2), t(0,3), t(0,4), t(1,5)\},$$
$$u_{15}(I) = \{t(0,2), t(0,4), t(1,5), t(3,6)\}.$$

Here are possible equations for $A(4,13)$, $A(4,15)$.

$$\begin{array}{ll} \mathrm{A(4,13)} & xyzw(x+y)(y+z)(z+w)(w+y)(y+z+w) \\ & (x+y+z)(x+y+w)(x+y+z+w)(x+2y+z+w) \end{array}$$

$$\begin{array}{ll} \mathrm{A(4,15)} & xyzw(x+y)(y+z)(z+w)(w+y) \\ & (x+y+z)(x+y+w)(y+z+w)(x+2y+z) \\ & (x+y+z+w)(x+2y+z+w)(x+2y+2z+w) \end{array}$$

The Poincaré polynomials[7] *of $A(3,7)$, $A(4,13)$, $A(4,15)$ are, respectively,* $[1][3]^2$, $[1][3][4][5]$, $[1][4][5]^2$. □

[7] See [O, section 2.3] for the definition of the Poincaré polynomial. We use the notation $[n] = 1 + nt$.

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Daan Krammer, University of Warwick, Mathematics Department, Coventry CV4 7AL, UK, `D.Krammer@warwick.ac.uk` .